On the Order of Magnitude of the Coefficients of a Fourier Series.

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§ 1. Riemann's theorem that the coefficients of a Fourier series converge to zero was shown by Lebesgue to still hold when integration is understood to be in the general sense now employed, absolutely convergent, or Lebesgue integration. Little progress has, however, been made in the determination of the order of magnitude of the coefficients. It has, indeed, been proved that, when the function has bounded variation, na_n and nb_n are bounded functions of n, and that, when the function is a continuous function of such a type as satisfies a condition of Lipschitz,[*] $n^q a_n$ and $n^q b_n$ converge to zero, where q is a positive number not greater than unity, depending on the particular Lipschitz condition satisfied by the function. As regards the second of these results, involving the satisfying of a condition of Lipschitz, it is to be remarked that, in well-known series of the type

$$\Sigma n^{-q} \cos nx \qquad \text{and} \qquad \Sigma n^{-q} \sin nx,$$

the functions of which they are the Fourier series do not, in any interval containing the origin, satisfy any condition of Lipschitz, being, indeed, unbounded.

In the present communication I obtain a number of theorems corresponding to each of these two results, including them as particular cases, and, at the same time, leading to the known properties of the simple sine and cosine series above referred to.

It will be remarked that the conditions supposed to hold in the neighbourhood of the origin and the remaining part of the interval of periodicity are different; this is inevitable, if they are to be as wide as possible. It might be, perhaps, supposed *a priori* that conditions of a far more general type would be adequate; this is not, however, the case. It is still more unusual for na_n and nb_n to converge than it is for the Fourier series to converge. Indeed, if the Fourier series converges at the origin, and na_n converges to zero, the Fourier series must do more than ordinarily converge, it must converge to a Cesàro order of negative unity. A similar statement holds good of the convergence of a_n and b_n when multiplied by a function of n of lower order of magnitude than n. There is, however, a theory of Cesàro convergence of successions analogous to that for series, where we consider instead of the convergence of na_n, nb_n, $n^q a_n$, $n^q b_n$, the arithmetical integral, or fractional, means of these quantities.

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[*] H. Lebesgue, "Sur la représentation trigonométrique approchée des fonctions satisfaisant à une condition de Lipschitz," 'Bull. de la Soc. Math. de France,' vol. 38, pp. 1–27 (1910).

When this is done, more general results are possible. For example, if the Cesàro order is of a type greater than or equal to unity, only the nature of the function in a small neighbourhood of the origin matters, equally when we consider convergence or oscillation.

We can, however, no longer, in employing this theory, speak of the results obtained as referring to the order of magnitude of the coefficients of a Fourier series. I propose, however, as they have an important application in the theory of Fourier series, to consider these questions on a subsequent occasion.

For simplicity of expression, the theorems have been stated for functions which are either odd or even. It is hardly necessary to point out that the results are perfectly general; when $f(x)$ is not itself odd or even, $\frac{1}{2}[f(x)+f(-x)]$ is the even, and $\frac{1}{2}[f(x)-f(-x)]$ the odd, function which takes the place of $f(x)$ in the theorems, a_n and b_n being then the coefficients of $\cos nx$ and $\sin nx$ in the Fourier series of $f(x)$.

§ 2. For completeness the theorem about to be given and the following one are stated generally, though information as to the order of the coefficients a_n and b_n is only given by the latter half of each.*

THEOREM 1.—*If a_n be the typical coefficient of the Fourier series of an even function of bounded variation, then*

$$na_n \rightarrow 0,$$

when the convergence is taken in the Cesàro manner, of any positive type whatever; moreover, the convergence is ordinary convergence if the function be an integral in every interval not containing the origin.

Let $g(x)$ be the function in question, then

$$na_n = \frac{1}{\pi}\int_{-\pi}^{\pi} ng(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} g(x)\,d(\sin nx) = -\frac{2}{\pi}\int_0^{\pi}\sin nx\,dg(x)$$

$$= -\frac{2}{\pi}\int_e^{\pi}\sin nx\,dg(x) - \frac{2}{\pi}\int_0^{e}\sin nx\,dg(x).$$

Case 1. Let $g(x)$ be an integral in every interval not containing the origin; then by the Theorem of Riemann-Lebesgue the first term of the expression just written down approaches zero.

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* It should be remarked that the second half of Theorem 1, if we assume the known properties of a function of bounded variation, reduces to the theorem which I have called that of Riemann-Lebesgue. It is inserted to preserve the parallelism between the Fourier series and its allied series. The whole of Theorem 2 breaks new ground.

Case 2. If $g(x)$ be not such an integral, then the first term approaches zero, when the convergence is taken in the Cesàro manner, with any positive index, as is easily seen,[*] but does not do so in the ordinary manner.

In either case we reduce to the consideration of the second term. This is numerically $\leq \frac{2}{\pi}\int_0^e \left| dg(x) \right|$; and therefore is as small as we please, if $g(x)$ is continuous at the origin.

But even if $g(x)$ is discontinuous at the origin, such a discontinuity could not affect this fact, since $\sin nx$ is zero at the origin. Therefore in all cases the second term in our expression approaches zero, which proves the theorem.

THEOREM 2.—*If b_n be the typical coefficient of the Fourier series of an odd function of bounded variation, then*

$$nb_n \to \frac{1}{\pi}(g(+0)-g(-0)) = \frac{1}{\pi}\,(\textit{jump at the origin}),$$

when the convergence is taken in the Cesàro manner, of any positive type whatever; moreover, the convergence is ordinary convergence if the function be an integral in every interval not containing the origin.

Let $g(x)$ be the odd function of bounded variation. Its Fourier series is unaffected by the values of $g(x)$ at the origin and the extremities π and $-\pi$, and we may therefore with advantage suppose the value of $g(x)$ at these points to be that given by summing the Fourier series of $g(x)$, namely zero, so that

$$g(0) = g(\pi) = g(-\pi) = 0.$$

We then have, if b_n be the typical Fourier coefficient of $g(x)$,

$$\begin{aligned} nb_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} ng(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi} n(g(x)-g(-x))\sin nx\,dx \\ &= -\frac{2}{\pi}\int_0^{\pi} g(x)\,d(\cos nx) = \frac{2}{\pi}\int_0^{\pi}\cos nx\,dg(x) \\ &= \frac{2}{\pi}\int_e^{\pi}\cos nx\,dg(x) + \frac{2}{\pi}\int_0^{e}\cos nx\,dg(x). \end{aligned}$$

Case 1. Let $g(x)$ be an integral in every interval not containing the origin; then the first term in the expression just obtained approaches zero as $n \to \infty$, by the Theorem of Riemann-Lebesgue.

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[*] In fact $\Sigma \sin nx$ oscillates boundedly for $e \leq x \leq \pi$, and therefore $\frac{1}{n^k}\sum_1^n \sin nx \to 0$ boundedly when $k>0$. It follows that $\sum_1^n \sin nx \to 0$ boundedly, if summed in the Cesàro manner, index k. This bounded sequence can by a known theorem be then integrated with respect to $g(x)$ term-by-term.

Case 2. Let $g(x)$ not be such an integral, then the first term approaches zero, as $n \to \infty$, provided the convergence be taken in any Cesàro manner, index > 0.[*] Thus only the second term needs consideration. But this differs from $2g(+0)/\pi$ by a quantity which is numerically $\leq 2(V(e)-V(+0))/\pi$, where $V(x)$ is the total variation of $g(x)$, by a quantity which is as small as we please, e being chosen sufficiently small. Thus

$$nb_n = 2g(+0)/\pi = (g(+0)-g(+0))/\pi = \frac{1}{\pi}\,(\textit{jump at the origin}),$$

the convergence being in the Cesàro or ordinary manner, according as we have to do with Case 1 or Case 2. This proves the theorem.

§ 4. The former of the two preceding theorems enables us to prove immediately that the derived series of the Fourier series of a function of bounded variation converges $(C, k-1)$, $(0>k)$; more generally it converges with any negative Cesàro type on the positive side of -1. The latter of our theorems gives us the corresponding theorem for the convergence of the allied series, involving the usual further condition.[†] The two theorems also show that, when $g(x)$ is an integral in every interval not containing the origin, the convergence is $(C, -1)$.

In the second half of Theorems 1 and 2[‡] we may, both in the enunciation and in the proof, avoid the concept of function of bounded variation by changing the condition that the function should have bounded variation into what is in our case the equivalent condition that the differential coefficient of the function should possess an absolutely convergent integral.

We divide the interval $(0, e)$ at the point $2\pi/n$. We then have

$$\int_0^{2\pi/n} f(x)\, n \,\overset{\cos}{\underset{\sin}{\cdot}}\, nx\,dx = \int_0^{2\pi} f(x/n) \,\overset{\cos}{\underset{\sin}{\cdot}}\, x\,dx \to \int_0^{2\pi} f(+0) \,\overset{\cos}{\underset{\sin}{\cdot}}\, x\,dx \to 0,$$

since, for all points x of the completely open interval $(0, \pi)$

$$f(\pi)-f(x) = \int_x^{\pi} f'(x)\,dx,$$

$f(x)$ being by hypothesis an integral. Indeed the right-hand side having by hypothesis a unique finite limit when $x \to 0$, the same is true of the left-hand side, thus $f(+0)$ exists and is finite. Also by the same equation $f(x)$ approaches this limit boundedly, since the right-hand side is numerically

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[*] This follows from the fact that $\Sigma \cos nx$ oscillates boundedly for $e \leq x \leq \pi$, and the rest of the argument is the same as for Theorem 1.

[†] See a forthcoming paper by the author. Cesàro summability of negative integral indices has apparently been regarded as offering difficulties of at least an inconvenient character. In the theory of Fourier series this concept seems to me to be as indispensable as that of ordinary convergence.

[‡] So far as they relate to ordinary convergence of na_n and nb_n.

$$\leqq \int_0^\pi \left| f'(x) \right| dx.$$

Again

$$\int_{2\pi/n}^{e} f(x)\, n \begin{matrix}\cos \\ \sin\end{matrix} nx\, dx = \left[f(x) \begin{matrix}\sin \\ 1-\cos\end{matrix} nx \right]_{2\pi/n}^{e} - \int_{2\pi/n}^{e} f' \begin{matrix}\sin \\ 1-\cos\end{matrix} nx\, dx,$$

where the last integral is numerically

$$\leqq \int_0^e \left| f'(x) \right| dx,$$

which is as small as we please, e being chosen conveniently small.

Hence, as before, the required results at once follow.

§ 5. In the next two theorems the assumptions made render $f(x)$ an absolutely convergent integral in every interval not containing the origin. The following simple lemma is used in the proofs.

LEMMA.—*If $f(x)$ is an absolutely convergent integral in the interval (e, π), where $0 < e$, then*

$$\mathop{\mathrm{Lt}}_{n\to\infty} \left\{ \int_e^\pi f(x)\, n \cos nx + f(ne) \sin ne \right\} = 0$$

and

$$\mathop{\mathrm{Lt}}_{n\to\infty} \left\{ \int_e^\pi f(x)\, n \sin nx - f(ne) \cos ne \right\} = 0,$$

provided in the latter case $f(\pi) = 0$, e.g., if $f(x)$ is an odd function represented at $x = \pi$ by its Fourier series.

We have in fact only to integrate by parts, and use the Theorem of Riemann-Lebesgue, by which $\int_e^\pi f'(x) \begin{matrix}\sin \\ \cos\end{matrix} nx\, dx$ has the limit zero.

We now have the following theorem:—

THEOREM 3.—*If $f(x)$ is an even function, which in a certain neighbourhood $(-e, e)$ surrounding the origin is such that $x\dfrac{df}{dx}$ is a function of bounded variation, and in the remaining part of the interval $(-\pi, \pi)$ is itself an absolutely convergent integral, then the coefficients of the derived series of the Fourier series of $f(x)$ converge to a unique limit, namely $\mathop{\mathrm{Lt}}\limits_{x\to 0}\left(-x\dfrac{df}{dx}\right)$.*

If a_n is the typical Fourier coefficient of $f(x)$, we have

$$\tfrac{1}{2}\pi\, na_n = \int_0^\pi f(x)\, n \cos nx\, dx = \mathop{\mathrm{Lt}}_{t\to 0} \int_t^e f(x)\, n \cos nx\, dx + \int_e^\pi f(x)\, n \cos nx\, dx$$

$$= \int_e^\pi f(x)\, n \cos nx + f(ne) \sin ne - \mathop{\mathrm{Lt}}_{t\to 0} \left(f(t) \sin nt + \int_t^e f'(t) \sin nt\, dt \right), \quad (1)$$

since $f(t)$ is the integral of $f'(t)$, in the interval (t, e), for in this interval, not containing the origin, $f'(t)$ is bounded.

Now if we write $g(x)$ for $xf'(x)$, we have

$$\left| f(e) - f(t) \right| = \left| \int_t^e f'(t)\,dt \right| = \left| \int_t^e \frac{g(t)}{t}\,dt \right| \leq \mathrm{A}\,(\log e - \log t),$$

where A is the upper bound of $|g(x)|$ in $(0, e)$. Therefore $t\,f(t)$ certainly approaches zero as t approaches zero, and therefore the same is true of $f(t)\sin nt$, n being for the moment fixed. Thus the limit on the right of (1) reduces to $\int_0^e f'(t)\sin nt\,dt$.

Using the Lemma (§ 5), we have therefore, by (1),*

$$\underset{n\to\infty}{\mathrm{Lt}}\, \tfrac{1}{2}\pi n a_n = -\underset{n\to\infty}{\mathrm{Lt}} \int_0^e xf'(x)\frac{\sin nx}{x}\,dx = -\underset{n\to\infty}{\mathrm{Lt}} \int_0^e g(x)\frac{\sin nx}{x}\,dx$$

$$= -\underset{n\to\infty}{\mathrm{Lt}} \int_0^e g(x)\,n\phi'(nx)\,dx, \qquad (2)$$

where
$$\phi(t) = \int_0^t \frac{\sin t}{t}\,dt,$$

and is therefore a bounded function of t in the whole infinite interval $(0, \infty)$. Now $\phi(x)$ being an absolutely convergent integral in any finite interval, and $g(x)$ a function of bounded variation, we may integrate by parts and write

$$\underset{n\to\infty}{\mathrm{Lt}}\, \tfrac{1}{2}\pi\, n a_n = \underset{n\to\infty}{\mathrm{Lt}} \left\{ -g(e)\,\phi(ne) + \int_0^e \phi(nx)\,dg(x) \right\}.$$

But as $n\to\infty$, $\phi(nx)\to\frac{1}{2}\pi$, except at $x = 0$, where its value remains zero, and the approach of $\phi(nx)$ to its limit is bounded. Thus by a known theorem in the theory of integration with respect to a function of bounded variation,

$$\underset{n\to\infty}{\mathrm{Lt}}\, \tfrac{1}{2}\pi\, n a_n = -g(e)\,\tfrac{1}{2}\pi + \underset{n\to\infty}{\mathrm{Lt}} \int_0^e \phi(nx)\,dg(x)$$

$$= -g(e)\,\tfrac{1}{2}\pi + \int_{+0}^e \tfrac{1}{2}\pi\,dg(x) + \int_0^{+0} 0\,dg(x) = \tfrac{1}{2}\pi\,g(+0);$$

thus
$$\underset{n\to\infty}{\mathrm{Lt}}\, n a_n = g(+0)$$

which proves the theorem.

COR.—*If $f(x)$ is an even function, such that $x\,(df/dx)$ is a function of bounded variation throughout the whole interval $(-\pi, \pi)$, then the theorem is true.*

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* The integrals in (2) are Lebesgue integrals, since $g(x)$ has an absolutely convergent integral from 0 to e, and $\sin nx/x$ is continuous.

§ 7. It will be noticed that, from and after the equation (2), in which the absolutely convergent integral $\frac{1}{\pi}\int_0^e g(x)\frac{\sin nx}{x}dx$ occurs, the proof consists merely in proving that this has the unique limit $g(+)0$. Neglecting quantities which vanish when n increases indefinitely, this is the expression for the sum of n terms of the Fourier series of $g(x)$; we might, therefore, have simply quoted Dirichlet's results here, and omitted the rest of the proof. I preferred, however, to give what is virtually a new proof of this theorem of Dirichlet's.

The proof of the following theorem has been written out slightly differently from that of the preceding theorem. Had we pursued the same lines, $(1-\cos nx)$ would have taken the place of sin nx in the preceding article, in this sense, that we should have reduced the discussion to that of the absolutely convergent integral $\frac{1}{\pi}\int_0^e g(x)\frac{1-\cos nx}{x}dx$, and accordingly have proved that nb_n is effectively the sum of the first n terms of the allied series of the Fourier series of $g(x)$. We have, accordingly, in the course of the following proof shown incidentally that the allied series of a function $g(x)$ of bounded variation converges to $\frac{1}{\pi}\underset{\epsilon\to 0}{\mathrm{Lt}}\int_\epsilon^\pi \frac{g(x)}{x}dx$, provided only this limit exists. This result had already been obtained by me by another method.*

§ 8. Analogous to the theorem of §6, we have the following:—

THEOREM 4.—*If $f(x)$ is an odd function having a definite limit as $x\to 0$, and such that, in a certain neighbourhood $(-e, e)$ surrounding the origin, $x(df/dx)$ is a function of bounded variation, while in the remaining part of the interval $(-\pi, \pi)$, $f(x)$ is itself an absolutely convergent integral, then the coefficients of the derived series of the Fourier series of $f'(x)$ converge to a unique limit, namely,*

$$\underset{n\to\infty}{\mathrm{Lt}}(nb_n) = \frac{1}{\pi}[f(+0)-f(-0)].$$

In fact, writing K for $2k\pi/n$, where k, like n, is an integer,

$$\tfrac{1}{2}\pi nb_n = \int_0^\pi f(x)n\sin nx\,dx = \int_0^K f(x)n\sin nx\,dx + \int_K^\pi f(x)n\sin nx\,dx, \quad (1)$$

and

$$\int_0^K f(x)n\sin nx\,dx = \int_0^{2k\pi} f(t/n)\sin t\,dt \to \int_0^{2k\pi} f(+0)\sin x\,dx \to 0. \quad (2)$$

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* "Konvergenzbedingungen für die verwandte Reihe einer Fourierschen Reihe," 'Münchener Bericht,' June 10, 1911.

Also

$$\int_K^\pi f(x)\, n \sin nx\, dx$$

$$= \int_e^\pi f(x)\, n \sin nx\, dx + \int_K^e f(x)\, d(1-\cos nx),$$

$$= \int_e^\pi f(x)\, n \sin nx\, dx + \Big[f(x)(1-\cos nx) \Big]_K^e - \int_K^e x f'(x) \frac{1-\cos nx}{x}\, dx,$$

$$= \left\{ \int_e^\pi f(x)\, n \sin nx\, dx - f(e) \cos ne \right\} + f(\mathrm{K}) + \int_K^e g(x) \frac{\cos nx}{x}\, dx, \qquad (3)$$

since $f(x)$ is the integral of $f'(x)$, in the interval (K, e) not containing the origin, for in this interval $f'(x)$ is bounded. Here

$$g(x) = xf'(x)$$

is a function of bounded variation. Putting

$$\phi(t) = \int_{2k\pi}^{t} \frac{\cos t}{t}\, dt,$$

so that $\phi(t)$ is an integral, we may therefore integrate the last term of (3) by parts, and write

$$\int_K^e g(x) \frac{\cos nx}{x}\, dx = \Big[g(x)\, \phi(nx) \Big]_K^e - \int_K^e \phi(nx)\, dg(x).$$

This is numerically $\leq \frac{1}{k\pi} \left(\left| g(e) \right| + \int_K^e \left| dg(x) \right| \right) \leq \frac{1}{k\pi} 2\mathrm{V}$, where V is the total variation of $g(x)$ from 0 to e, and may therefore be put $= \mathrm{M}/k$, where $|\mathrm{M}| \leq 2\mathrm{V}/\pi$.

Hence, using the Lemma (§ 5), the right-hand side of (3) differs from $f(\mathrm{K}) + \mathrm{M}/k$ by a quantity which vanishes when $n \to \infty$, and differs therefore from $f(+0) + \mathrm{M}/k$ by a quantity which vanishes when $n \to \infty$. Thus by (1), (2), and (3).

$$\tfrac{1}{2}\pi n b_n - f(+0)$$

remains, as $n \to \infty$, numerically $\leq 2\mathrm{V}/\pi k$. But k may be chosen as large as we please, therefore, $\frac{1}{2}\pi n b_n - f(+0)$ can have no other limit except zero, when $n \to \infty$. This proves the theorem.

§ 9. Always provided that we ensure the vanishing in the limit of the portion of our expression due to the interval (e, π) in the Theorems 3 and 4, the problem of the order of magnitude thus reduces to that of determining the conditions of convergence of the Fourier series of $x\, df/dx$* and of its allied series. Corresponding to every known theorem regarding the convergence of such series, we have accordingly theorems relating to the order of magnitude of na_n and nb_n.

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* This function must accordingly possess an absolutely convergent integral in $(-\pi, \pi)$. It will be immediately recognised that this is the case in the theorems given below.

With regard to the coefficient b_n, it should be remarked that the extra condition for the convergence of the allied series, namely, the existence of the limit of a certain integral, reduces in the case of the order of magnitude problems to the condition that $f(x)$ should have a unique limit as $x \to 0$.

Thus we have the following theorems which correspond to a test due to de la Vallée Poussin for the Fourier series, and my counterpart of it for the allied series.

THEOREM 5.—*If $f(x)$ is an even function, and such that in a certain neighbourhood surrounding the origin*

$$\frac{1}{x}\int_0^x x\frac{df}{dx}\,dx$$

is a function of bounded variation, while in every interval not containing the origin $f(x)$ is itself an absolutely convergent integral, then

$$\operatorname*{Lt}_{n\to\infty} na_n = \operatorname*{Lt}_{x\to 0}\left(-\frac{1}{x}\int_0^x x\frac{df}{dx}\,dx\right).$$

THEOREM 6.—*If $f(x)$ is an odd function satisfying the same conditions, and the further condition that $f(+0)$ exists, then*

$$\operatorname*{Lt}_{n\to\infty} nb_n = \frac{1}{\pi}\left(f(+0)-f(-0)\right).$$

Again, corresponding to tests due to myself, which I have recently simplified, we have the theorems:

THEOREM 7.—*If $f(x)$ is an even function, and such that in a certain neighbourhood surrounding the origin*

$$x\frac{df}{dx} = \frac{1}{x}\int_0^x h(x)\,dx,$$

where $\frac{1}{x}\int_0^x |h(x)|\,dx$ is bounded, while in the remaining part of the interval $(-\pi, \pi)$, $f(x)$ is itself an absolutely convergent integral, then

$$\operatorname*{Lt}_{n\to\infty} na_n = \operatorname*{Lt}_{x\to 0}\left(-x\frac{df}{dx}\right),$$

this limit being supposed to exist.

THEOREM 8.—*If $f(x)$ is an odd function, satisfying the same conditions as in Theorem 7, and satisfying the further condition that $f(+0)$ exists, then*

$$\operatorname*{Lt}_{n\to\infty} nb_n = \frac{1}{\pi}\left[f(+0)-f(-0)\right].$$

The conditions of Theorems 7 and 8 evidently render $f(x)$ an absolutely convergent integral in every interval not containing the origin. Thus all that is necessary to ensure that the argument employed in Theorems 3 and 4, and explained in § 7, is applicable is to show that, in all the theorems of the present article, $xf(x)$ converges to zero as $x \to 0$.

In the cases of Theorems 5 and 6 we have

$$\int_t^e x\frac{df}{dx}\,dx = ef(e)-tf(t)-\int_t^e f(x)\,dx$$

and therefore, since $x\,df/dx$ and $f(x)$ are both summable in the whole interval $(0, e)$, by hypothesis, $tf(t)$ has a unique finite limit as $t \to 0$. Now this limit cannot be other than zero, for if it were, $f(x)$ would in the neighbourhood of the origin be of the order A/t, and therefore not summable from 0 to e, whereas the Fourier series considered is that of $f(x)$, which is accordingly summable from $-\pi$ to π.

In the cases of Theorems 7 and 8, $x\,df/dx$ is bounded and therefore possesses an absolutely convergent integral in $(0, e)$. Also from the identity

$$\frac{d}{dx}[xf(x)] = f(x)+x\,df/dx,$$

we deduce

$$ef(e)-tf(t) = \int_t^e f(x)\,dx+\int_t^e x\frac{df}{dx}\,dx,$$

since $xf(x)$ is an integral in (t, e). But each of the integrals on the right exists in the whole interval $(0, e)$, therefore the right-hand side of the last equation has a unique finite limit as $t \to 0$; hence also $tf(t)$ has a unique limit, which, as before, must be zero.

§ 10. It will be noticed that, in the next theorems, bounded functions have taken the place of the functions of bounded variation in Theorems 3 and 4. That we could not formerly make this more general assumption is due to the fact that there is a sudden change as p reaches zero when decreasing from unity. Otherwise the two theorems which follow are precisely analogous to Theorems 3 and 4, and constitute, like them, very wide generalisations of the properties of the familiar simple series of cosines or sines.

It should also be remarked that in this and in the following theorems up to Theorem 12, we write the generalised condition of Lipschitz briefly. The notation $O(h^{1-p})$ must be interpreted to mean

$$\lambda h^{1-p}, \qquad (0<\lambda;\ 0<p<1),$$

λ being a quantity which has finite bounds independent of x and h.

When λ is a quantity as small as we please, which occurs, for example, if the quantity p employed is not the smallest at our disposal (*e.g.*, when there is no smallest), we use o in place of O.

THEOREM 9.—*If $f(x)$ is an even function which in a certain neighbourhood $(-e, e)$ surrounding the origin is such that $x^{1+p}\, df/dx$ is a bounded function with a unique limit as $x \to 0$, and in the remaining part of the interval $(-\pi, \pi)$ itself satisfies the condition of Lipschitz,*

$$|f(x+h)-f(x)| = O(h^{1-p}), \qquad (0 < h;\ 0 < p < 1),$$

then

$$n^{1-p} a_n = O(1).$$

Moreover, if in the Lipschitz condition O *be replaced by* o,

$$n^{1-p} a_n \to \operatorname*{Lt}_{x \to 0} \cos \tfrac{1}{2} p\pi \left(-x^{1+p} f'(x)\right) / \Gamma(p+1).$$

Let us put

$$f(x) = f_1(x) + f_2(x),$$

where $f_1(x) = f(x) - f(e)$ in the interval $(-e \leq x \leq e)$, and is zero elsewhere.

Then $f_2(x)$ satisfies our condition of Lipschitz throughout the whole interval $(-\pi, \pi)$. Therefore by Lebesgue's result (*loc. cit.*, § 1 *supra*) the theorem holds when for f we substitute f_2 in the enunciation. Hence it is only necessary to prove that the theorem is true when for f we substitute f_1.

Now in the interval (e, π) we have $f_1(x) = 0$, and if $0 < t < e$, in (t, e) the function $f'_1(x) = f'(x)$ is bounded, therefore we have

$$f_1(t) = -\int_t^e f'(t)\, dt = -\int_t^e t^{-1-p} g(t)\, dt,$$

where $g(t)$ is the bounded function hypothecated in the enunciation. Therefore we may write

$$\begin{aligned} &\int_0^\pi f_1(x)\, n \cos nx\, dx \\ &\quad = \operatorname*{Lt}_{t \to 0} \int_t^e f_1(x)\, d(\sin nx) \\ &\quad = \operatorname*{Lt}_{t \to 0} \left(-f_1(t) \sin nt - \int_t^e f_1'(x) \sin nx\, dx \right) \\ &\quad = -\int_0^e g(x)\, x^{-1-p} \sin nx\, dx = -n^p \int_0^{ne} g(t/n)\, t^{-1-p} \sin t\, dt, \end{aligned}$$

since $\left| f_1(t) \right| \leq A \int_t^e t^{-1-p}\, dt$, where A is the upper bound of $|g(x)|$ in $(0, e)$, so that $t f_1(t)$, and therefore also $f_1(t) \sin nt$, for fixed n, vanish as $t \to 0$.

Now dividing the interval of integration $(0, ne)$ at the point $2k\pi$,

$$\left| \int_{2k\pi}^{ne} g(t/n)\, t^{-1-p} \sin t\, dt \right| \leq \frac{B}{p} \left[(2k\pi)^{-p} + (ne)^{-p} \right],$$

where B is the upper bound of $|g(x)|$ in $(0, e)$. Letting n increase indefinitely this term is therefore numerically $\leq B(2k\pi)^{-p}/p$, which is as small as we please, if k is chosen conveniently large.

Also since $g(t/n) \to g(+0)$ boundedly,

$$\int_0^{2k\pi} g(t/n)\, t^{-1-p} \sin t\, dt \to g(+0) \int_0^{2k\pi} t^{-1-p} \sin t\, dt.$$

Thus finally, letting first n and then k increase indefinitely,

$$\underset{n\to\infty}{\text{Lt}} \int_0^{\pi} f_1(x)\, n^{1-p} \cos nx\, dx = -g(+0) \int_0^{\infty} t^{-1-p} \sin t\, dt$$

$$= -g(+0)(\cos \tfrac{1}{2} p\pi)/\Gamma(p+1),$$

which proves the theorem in both its parts.

§ 11. Similarly we have the following theorem :—

THEOREM 10.—*If $f(x)$ is an odd function, which in a certain neighbourhood $(-e, e)$ surrounding the origin is such that $x^{1+p}(df/dx)$ is a bounded function with a unique limit as $x \to 0$, and in the remaining part of the interval $(-\pi, \pi)$ itself satisfies the condition of Lipschitz,*

$$f(x+h) - f(x) = O\phi(h^{1-p}), \qquad (0 < h;\ 0 < p < 1),$$

then $$n^{1-p} b_n = O(1).$$

Moreover, if in the Lipschitz condition O be replaced by o, then

$$n^{1-p} b_n \to \underset{x\to 0}{\text{Lt}} \sin \tfrac{1}{2} p\pi (-x^{1+p} f'(x)/\Gamma(p+1)).$$

It is, in fact, only necessary to follow the reasoning of the preceding article, $(1 - \cos nx)$ taking the place of $\sin nx$.

§ 12. We now proceed to give other generalisations of the properties of simple sine and cosine series referred to at the end of § 7. That we have convergence to zero is due to the fact that whatever power of x^{-p} we take, there is no index which will make this just summable. On the other hand, if a smaller index is used than $1/p$, the example of the simple sine and cosine series shows that we must get o and not O.

THEOREM 11.—*If $f(x)$ is an even function, which in $(-e, e)$ is such that $\left| x \dfrac{df}{dx} \right|^{1/p}$ is integrable (summable) and in the remaining part of the interval $(-\pi, \pi)$ itself satisfies the condition of Lipsehitz,*

$$f(x+h) - f(x) = O(h^{1-p}), \qquad (0 < h;\ 0 < p < 1),$$

then $$n^{1-p} a_n = O(1).$$

Moreover, if in the Lipschitz condition, O be replaced by o, then

$$n^{1-p} a_n \to 0.$$

In fact, writing $g(x)$ for $x\, df/dx$,

$$ef(e) - tf(t) = \int_t g(x)\, dx + \int_t^e f(x)\, dx;$$

and, therefore, as in the preceding article, $tf(t) \to 0$ as $t \to 0$.

Thus we get, as in the proof of Theorem 3, or rather in Theorems 5, 7, 9, in accordance with the remarks of § 9, that the convergence of na_n is virtually the same as that of $-\int_0^e g(x)(\sin nx/x)\,dx$.

We now use the theorem of § 8, p. 439 of my paper "On the Mode of Oscillation of a Fourier Series and of its Allied Series,"* which tells us that this latter integral when divided by n^p converges to zero. Hence na_n divided by n^p converges to zero, that is

$$n^{1-p}a_n \to 0.$$

THEOREM 12.—*If $f(x)$ is an odd function which satisfies the same conditions as in Theorem* 11,

$$n^{1-p}b_n \to 0.$$

The argument is precisely the same as that used in proving Theorem 11, except that $\int_0^e g(x)[(1-\cos nx)/x]\,dx$ takes the place of $\int_0^e g(x)(\sin nx/x)\,dx$.

Reasoning precisely analogous to that employed in the article quoted at the end of the preceding proof leads to the same order of magnitude of this integral as of the other integral, as is, indeed, pointed out in the article quoted, this integral being related to the allied series as the other is to the Fourier series.

§ 13. In the next two theorems we use, instead of the condition of Lipschitz, the condition of Lipschitz-Dini, namely,

$$\operatorname*{Lt}_{h\to 0}\left((\log\frac{1}{h})[f(x+h)-f(x)]\right) = 0,$$

the convergence being uniform for all values of x, or as we shall for brevity write it

$$|f(x+h)-f(x)| = o\left(\frac{1}{\log 1/h}\right).$$

THEOREM 13.—*If $f(x)$ is an even function, which in $(-e, e)$ is such that $x(df/dx)$ is bounded, and in the remaining part of the interval $(-\pi, \pi)$ itself satisfies the condition of Lipschitz-Dini,*

$$f(x+h)-f(x) = o\left(\frac{1}{\log 1/h}\right),$$

then
$$na_n = O\log n.$$

Further, if in addition $x(df/dx)$ has a unique finite limit as $x\to 0$, then

$$na_n = o\log n,$$

that is
$$na_n/\log n \to 0.$$

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* 'Lond. Math. Soc. Proc.,' Ser. 2, vol. 12.

The proof of this theorem is precisely similar to that of Theorem 9, using the corresponding result of Lebesgue's,[*] and § 3, p. 436 of my paper "On the Mode of Oscillation of a Fourier Series and of its Allied Series."

THEOREM 14.—*If $f(x)$ is an odd function, satisfying the same conditions as in Theorem* 13,

$$nb_n = O \log n,$$

and if, in addition, $x\,(df/dx)$ has a unique finite limit as $x \to 0$, then

$$\underset{n\to\infty}{\mathrm{Lt}}\,(nb_n/\log n) = 0.$$

Here, again, $(1-\cos nx)$ replaces $\sin nx$ in the part of the integral considered due to the interval $(0, e)$.

[*Added December,* 1916.—In a communication made to the Société Helvétique, in August, entitled "Sur les intégrales multiples et les séries de Fourier," I remarked that I had occupied myself with the extension to multiple Fourier series of results obtained by myself and others with respect to Fourier series of a single variable, and that I had as yet not found any which did not hold *mutatis mutandis* in the general case. This is also true of the results of the present paper.]

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[*] *Loc. cit.*, § 1 *supra.*